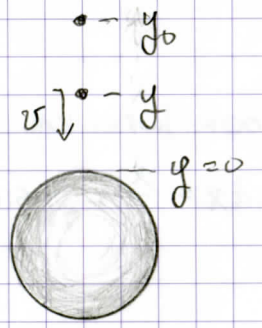
TM5 Pr 5.5

A PARTICLE FALLS TO THE EARTH STARTING FROM REST AT A GREAT HEIGHT. NEGLECT AIR RESISTANCE AND SHOW THAT THE PARTICLE REQUIRES APPROXIMATELY 9/11 OF THE TOTAL TIME OF FALL TO TRAVERSE THE FIRST HALF OF THE DISTANCE.



APPLY NSL:

$$\Sigma F = mv\frac{dv}{dy}$$

$$-\frac{GMm}{y^2} = mv\frac{dv}{dy}$$

SEPARATE VARIABLES

$$-\int_{y_0}^{y}\frac{GM}{y^2}\,dy = \int_0^v v\,dv \qquad \frac{d}{dy}\frac{1}{y} = -\frac{1}{y^2}$$

INTEGRATING

$$+GM\left(\frac{1}{y}-\frac{1}{y_0}\right) = \frac{1}{2}v^2$$

→ SINCE v IS DOWN AND y IS UP, v MUST BE NEGATIVE SO TAKE THE "−" FOR THE $(\pm x)^2 = x^2$

$$\Rightarrow -\sqrt{2GM\left(\frac{1}{y}-\frac{1}{y_0}\right)} = v = -\frac{dy}{dt}$$

SEPARATE VARIABLES

$$\int_0^t dt = \int_{y_0}^{y}\frac{-dy}{\sqrt{2GM\left(\frac{1}{y}-\frac{1}{y_0}\right)}}$$

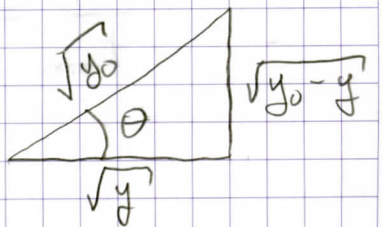
$$t = \int_{y_0}^{y}\frac{-\sqrt{y}\,dy}{\sqrt{2GM\left(1-\frac{y}{y_0}\right)}}$$

SUBSTITUTE

$$y = y_0\cos^2\theta$$

$$dy = -2y_0\cos\theta\sin\theta$$

$$\frac{y}{y_0} = \cos^2\theta$$



$$= \int_0^{\theta}\frac{+\sqrt{y_0\cos^2\theta}\,(2y_0\cos\theta\sin\theta)\,d\theta}{\sqrt{2GM(1-\cos^2\theta)}}$$

$y = y_0 \Rightarrow \theta = 0$

$y = 0 \Rightarrow \theta = \frac{\pi}{2}$

→

$$t = \int_0^{\theta} \frac{2 y_0^{3/2} \cos^2\theta \sin\theta}{\sqrt{2GM}\sin\theta}\, d\theta$$

$$t = \frac{2 y_0^{3/2}}{\sqrt{2GM}} \int_0^{\theta} \cos^2\theta \, d\theta$$

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$$\int \cos^2 ax\, dx = \frac{x}{2} + \frac{1}{4a}\sin 2ax$$

$$t = \frac{2 y_0^{3/2}}{\sqrt{2GM}} \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\theta}$$

$$t = \frac{y_0^{3/2}}{\sqrt{2GM}} \left(\theta + \frac{1}{2}\sin 2\theta \right)$$

FIND $t_{1/2}$: $y = \frac{1}{2} y_0 \Rightarrow \cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$$t_{1/2} = \frac{y_0^{3/2}}{\sqrt{2GM}} \left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} \right) = \frac{y_0^{3/2}}{\sqrt{2GM}} \left(\frac{\pi}{4} + \frac{1}{2} \right) = t_{1/2}$$

FIND t_{TOTAL}: $y = 0 \Rightarrow \cos^2\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

$$t_{1/2} = \frac{y_0^{3/2}}{\sqrt{2GM}} \left(\frac{\pi}{2} + \frac{1}{2}\sin\pi \right) = \frac{y_0^{3/2}}{\sqrt{2GM}} \frac{\pi}{2} = t_{1/2}$$

FIND THE RATIO:

$$\frac{t_{1/2}}{t_{TOTAL}} = \frac{\frac{\pi}{4} + \frac{1}{2}}{\frac{\pi}{2}} = \frac{\frac{\pi}{2} + 1}{\pi} \approx 0.818 \approx \boxed{\frac{9}{11} = \frac{t_{1/2}}{t_{TOTAL}}}$$

NOTE BOOK